

# ON THE OPPENHEIM'S "FACTORISATION NUMERORUM" FUNCTION

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## 1. INTRODUCTION

Let  $f(n)$  denote the number of distinct unordered factorisations of the natural number  $n$  into factors larger than 1. For example,  $f(28) = 4$  as 28 has the following factorisations

$$28, 2 \cdot 14, 4 \cdot 7, 2 \cdot 2 \cdot 7.$$

In this paper, we address three aspects of the function  $f(n)$ . For the first aspect, in [1], Canfield, Erdős and Pomerance mention without proof that the number of values of  $f(n)$  that do not exceed  $x$  is  $x^{o(1)}$  as  $x \rightarrow \infty$ . Our first theorem in this note makes this result explicit.

For a set  $\mathcal{A}$  of positive integers we put  $\mathcal{A}(x) = \{n \in \mathcal{A} : n \leq x\}$ .

**Theorem 1.** *Let  $\mathcal{A} = \{f(m) : m \in \mathbb{N}\}$ . Then*

$$\#\mathcal{A}(x) = x^{O(\log \log \log x / \log \log x)}, \quad \text{as } x \rightarrow \infty.$$

Secondly, there is a large body of literature addressing average values of various arithmetic functions in short intervals. Our next result gives a lower bound for the average of  $f(n)$  over a short interval.

**Theorem 2.** *Uniformly for  $x \geq 1$  and  $y > e^{e^e}$ , we have*

$$\frac{1}{y} \sum_{x \leq n \leq x+y} f(n) \geq \exp \left( \left( \frac{4}{\sqrt{2e}} + O \left( \frac{(\log \log \log y)^2}{\log \log y} \right) \right) \frac{\sqrt{\log y}}{\log \log y} \right).$$

Finally, there are also several results addressing the behavior of positive integers  $n$  which are multiples of some other arithmetic function of  $n$ . See, for example, [3], [5], [9] and [10] for problems related to counting positive integers  $n$  which are divisible by either  $\omega(n)$ ,  $\Omega(n)$  or  $\tau(n)$ , where these functions are the number of distinct prime factors of  $n$ , the number of total prime factors of  $n$ , and number of divisors of  $n$ , respectively. Our next and last result gives an upper bound on the counting function of the set of positive integers  $n$  which are multiples of  $f(n)$ .

**Theorem 3.** *Let  $\mathcal{B} = \{n : f(n) \mid n\}$ . Then*

$$\#\mathcal{B}(x) = \frac{x}{(\log x)^{1+o(1)}} , \quad \text{as } x \rightarrow \infty.$$

## 2. PRELIMINARIES AND LEMMAS

The function  $f(n)$  is related to various partition functions. For example,  $f(2^n) = p(n)$ , where  $p(n)$  is the number of partitions of  $n$ . Furthermore,  $f(p_1 p_2 \cdots p_k) = B_k$ , where  $B_k$  is the  $k$ th Bell number which counts the number of partitions of a set with  $k$  elements in nonempty disjoint subsets. In general,  $f(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k})$  is the number of partitions of a multiset consisting of  $\alpha_i$  copies of  $\{i\}$  for each  $i = 1, \dots, k$ . Throughout the paper, we put  $\log x$  for the natural logarithm of  $x$ . We use  $p$  and  $q$  for prime numbers and  $O$  and  $o$  for the Landau symbols.

The following asymptotic formula for the  $k$ th Bell number is due to de Bruijn [4].

**Lemma 1.**

$$\frac{\log B_k}{k} = \log k - \log \log k - 1 + \frac{\log \log k}{\log k} + \frac{1}{\log k} + O\left(\frac{(\log \log k)^2}{(\log k)^2}\right).$$

We also need the Stirling numbers of the second kind  $S(k, l)$  which count the number of partitions of a  $k$  element set into  $l$  nonempty disjoint subsets. Clearly,

$$(1) \quad B_k = \sum_{l=1}^k S(k, l).$$

We now formulate and prove a few lemmas about the function  $f(n)$  which will come in handy later on.

The first lemma is an easy observation, so we state it without proof.

**Lemma 2.** *If  $a \mid b$ , then  $f(a) \leq f(b)$ .*

We let  $p_n$  denote the  $n$ th prime number and  $\alpha_1(n)$  denote the maximal exponent of a prime appearing in the prime factorization of  $n$ . Let  $n$  be a positive integer with prime factorization

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k},$$

where  $q_1, \dots, q_k$  are distinct primes and  $\alpha_1(n) := \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$ . We put  $n_0 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ , and observe that  $f(n) = f(n_0)$ . This observation will play a crucial role in the proof of Theorem 1.

The following lemma gives upper bounds for  $\alpha_1(n)$  and  $\omega(n)$  when  $f(n) \leq x$ .

**Lemma 3.** *Let  $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k}$ , where  $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$  and  $f(n) \leq x$ . Then*

- (i)  $\alpha_1 = O((\log x)^2)$ ;
- (ii)  $k = \omega(n) = O(\log x / \log \log x)$ .

*Proof.* It follows from Lemma 2 that

$$f(n) \geq f(q_1^{\alpha_1}) = p(\alpha_1).$$

Using the following asymptotic formula for  $p(n)$  due to Hardy and Ramanujan [6]

$$(2) \quad p(n) \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3}),$$

we conclude that  $\exp(c\sqrt{\alpha_1}) \leq x$  holds with some constant  $c > 0$ . Hence, (i) follows. In order to prove (ii), let again  $n_0 = p_1 p_2 \cdots p_k$ . By Lemma 2, we have  $f(n_0) \leq f(n) \leq x$ . Furthermore,  $f(n_0) = B_k$ . It now follows from Lemma 1, that

$$\exp((1 + o(1))k \log k) = B_k \leq x,$$

as  $k \rightarrow \infty$ , yielding

$$k = O\left(\frac{\log x}{\log \log x}\right),$$

which completes the proof of the lemma.  $\square$

Recall that the Möbius function  $\mu(m)$  of the positive integer  $m$  is  $(-1)^{\omega(m)}$  if  $m$  is squarefree and 0 otherwise.

For a positive integer  $k$  and positive real numbers  $A \leq B$  we let

$$\mathcal{M}_{k,A,B} = \{m : \mu(m) \neq 0, \omega(m) = k, \text{ if } p \mid m \text{ then } p \in [A, B]\}.$$

We also put

$$S_{A,B} = \sum_{A \leq p \leq B} \frac{1}{p}.$$

**Lemma 4.** *Uniformly in  $A \geq 2$ ,  $B \geq 3$  and  $k \geq 2$ , we have*

$$\sum_{m \in \mathcal{M}_{k,A,B}} \frac{1}{m} \geq \left(1 + O\left(\frac{k^2}{S_{A,B}^2 A \log A}\right)\right) \frac{1}{k!} S_{A,B}^k.$$

*Proof.* We omit the dependence of the subscripts in order to simplify the presentation. It is not hard to see that

$$(3) \quad \sum_{m \in \mathcal{M}} \frac{1}{m} \geq \frac{1}{k!} \left( \sum_{A \leq p \leq B} \frac{1}{p} \right)^k - \sum_{A \leq p \leq B} \frac{1}{p^2} \frac{1}{(k-2)!} \left( \sum_{A \leq p \leq B} \frac{1}{p} \right)^{k-2}.$$

Indeed, if  $m = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$ , with  $\alpha_1 \geq 2$  and  $\alpha_1 + \cdots + \alpha_s = k$ , then, by unique factorization, in the first sum on the right hand side of inequality (3), the number  $1/m$  appears with coefficient

$$\frac{1}{k!} \left( \frac{k!}{\alpha_1! \cdots \alpha_s!} \right) = \frac{1}{\alpha_1! \cdots \alpha_s!},$$

while in the second sum in the right hand side of the inequality (3), the number  $1/m$  appears with coefficient

$$\sum_{\substack{1 \leq i \leq s \\ \alpha_i \geq 2}} \frac{1}{(k-2)!} \left( \frac{(k-2)!}{\alpha_1! \cdots (\alpha_i-2)! \cdots \alpha_s!} \right) > \frac{1}{\alpha_1! \cdots \alpha_s!}.$$

This establishes inequality (3). Using this inequality, we get

$$\begin{aligned} \sum_{m \in \mathcal{M}} \frac{1}{m} &\geq \frac{S^k}{k!} - \frac{1}{(k-2)!} S^{k-2} \sum_{A \leq p \leq B} \frac{1}{p^2} \\ &\geq \frac{S^k}{k!} \left( 1 - \frac{k^2}{S^2} \sum_{p \geq A} \frac{1}{p^2} \right). \end{aligned}$$

An argument involving the Prime Number Theorem and partial summation gives

$$\sum_{p \geq A} \frac{1}{p^2} = O\left(\frac{1}{A \log A}\right).$$

Hence,

$$\sum_{m \in \mathcal{M}} \frac{1}{m} \geq \frac{S^k}{k!} \left( 1 + O\left(\frac{k^2}{S^2 A \log A}\right) \right).$$

This completes the proof of the lemma.

### 3. PROOFS OF THE THEOREMS

**3.1. Proof of Theorem 1.** For a positive integer  $n$ , we let again  $n_0$  and  $\alpha_1(n)$  be the functions defined earlier. We let  $\mathcal{A}(x) = \{m_1, \dots, m_t\}$  be such that  $m_1 < m_2 < \cdots < m_t$  and let  $\mathcal{N} = \{n_1, \dots, n_t\}$  be positive integers such that  $n_i$  is minimal among all positive integers  $n$  with  $f(n) = m_i$  for all  $i = 1, \dots, t$ . It is clear that if  $n \in \mathcal{N}$ , then  $n = n_0$ . Since  $\#\mathcal{A}(x) = t = \#\mathcal{N}$ , it suffices to bound the cardinality of  $\mathcal{N}$ .

We partition this set as  $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ , where

$$\begin{aligned} \mathcal{N}_1 &= \{n \in \mathcal{N} : \alpha_1(n) \leq \log \log x\}, \\ \mathcal{N}_2 &= \left\{ n \in \mathcal{N} : \omega(n) \leq \frac{\log x}{(\log \log x)^2} \right\}, \end{aligned}$$

and

$$\mathcal{N}_3 = \mathcal{N} \setminus \mathcal{N}_1 \cup \mathcal{N}_2.$$

If  $n \in \mathcal{N}_1$ , then  $n$  has at most  $O(\log x / \log \log x)$  prime factors (by Lemma 3), each one appearing at an exponent less than  $\log \log x$ .

Therefore,

$$\begin{aligned} \#\mathcal{N}_1 &= (\log \log x)^{O(\log x / \log \log x)} = \exp \left( O \left( \frac{\log x \log \log \log x}{\log \log x} \right) \right) \\ (4) \quad &= x^{O\left(\frac{\log \log \log x}{\log \log x}\right)} \end{aligned}$$

as  $x \rightarrow \infty$ .

Next, we observe that an integer in  $\mathcal{N}_2$  has at most  $\log x / (\log \log x)^2$  prime factors, each appearing at an exponent  $O((\log x)^2)$  (by Lemma 3). Thus,

$$\begin{aligned} \#\mathcal{N}_2 &\leq (O((\log x)^2))^{\log x / (\log \log x)^2} = \exp \left( \frac{(2 + o(1)) \log x}{\log \log x} \right) \\ (5) \quad &= x^{o\left(\frac{\log \log \log x}{\log \log x}\right)} \end{aligned}$$

as  $x \rightarrow \infty$ .

Finally, let  $n \in \mathcal{N}_3$ , and write it as

$$n = p_1^{\alpha_1} \cdots p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k},$$

where we put

$$i := \max\{j \leq k : \alpha_j \geq y\},$$

where  $y = \log \log x / \log \log \log x$ .

Observe that the divisor  $p_{i+1}^{\alpha_{i+1}} \cdots p_t^{\alpha_t}$  of  $n$  can be chosen in at most

$$(6) \quad (y+1)^k = (y+1)^{O(\log x / \log \log x)} = \exp \left( O \left( \frac{\log x \log \log \log x}{\log \log x} \right) \right)$$

ways. Furthermore, by Lemma 3, we trivially have that  $n' = p_1^{\alpha_1} \cdots p_i^{\alpha_i}$  can be chosen in at most

$$(O((\log x)^2))^i = \exp(O(i \log \log x)).$$

Thus, putting  $\mathcal{N}_4$  for the subset of  $\mathcal{N}_3$  such that  $i \leq \log x / (\log \log x)^2$ , we get that

$$(7) \quad \#\mathcal{N}_4 \leq \exp \left( O \left( \frac{\log x}{\log \log x} \right) \right).$$

From now on, we look at  $n \in \mathcal{N}_5 = \mathcal{N}_3 \setminus \mathcal{N}_4$ .

For each  $t$ , we let  $k_t$  be such that  $S(t, k_t)$  is maximal among the numbers  $S(t, k)$  for  $k = 1, \dots, t$ . By formula (1), the definition of  $k_t$ , and Lemma 1, we have that

$$S(t, k_t) \geq \frac{B_t}{t} = \frac{\exp((1 + o(1))t \log t)}{t} = \exp((1 + o(1))t \log t)$$

as  $t \rightarrow \infty$ . We now claim that

$$f(n) \geq f(n') \geq f((p_1 \cdots p_i)^y) \geq \frac{S(i, k_i)^y}{(yk_i)!}.$$

The first three inequalities follow immediately from Lemma 2, so let us prove the last one.

Note that  $S(i, k_i)$  counts the number of factorizations of  $p_1 p_2 \cdots p_i$  in precisely  $k_i$  factors. Therefore,  $(S(i, k_i))^y$  counts the number of factorizations of  $(p_1 p_2 \cdots p_i)^y$  into  $k_i y$  square-free factors, where we count each such factorization at most  $(k_i y)!$  times. This establishes the claim.

Since  $i$  tends to infinity for  $n \in \mathcal{N}_5$ , we get that

$$S(i, k_i)^y \geq \exp((1 + o(1))yi \log i).$$

Furthermore, we trivially have

$$(k_i y)! \leq (k_i y)^{k_i y} = \exp(k_i y \log(k_i y)).$$

Thus,

$$(8) \quad f(n) \geq \frac{S(i, k_i)^y}{(k_i y)!} \geq \exp((1 + o(1))yi \log i - k_i y \log(k_i y))$$

as  $x \rightarrow \infty$ . We next show that for our choices of  $y$  and  $i$  we have

$$k_i y \log(k_i y) = o(yi \log i) \quad \text{as} \quad x \rightarrow \infty.$$

Indeed, using the fact

$$k_i = (1 + o(1)) \frac{i}{\log i} \quad \text{as} \quad i \rightarrow \infty$$

(see, for example, [2]), we see that the above condition is equivalent to

$$\log y = o((\log i)^2),$$

which holds as  $x \rightarrow \infty$  because  $y = \log \log x / \log \log \log x$  and  $i > \log x / (\log \log x)^2$ . Now the inequality  $f(n) \leq x$  together with (8) and the fact that  $\log i \geq (1 + o(1)) \log \log x$  implies that

$$(9) \quad i \leq (1 + o(1)) \frac{\log x}{y \log \log x} \quad \text{as} \quad x \rightarrow \infty,$$

therefore  $n'$  can be chosen in at most

$$(10) \quad (O((\log x)^2))^i \leq (O((\log x)^2))^{(1+o(1))\frac{\log x}{y \log \log x}} = \exp\left((2+o(1))\frac{\log x}{y}\right)$$

ways. As we have already seen at (6), the complementary divisor  $n/n' = p_{i+1}^{\alpha_{i+1}} \cdots p_t^{\alpha_t}$  of  $n$  can be chosen in at most

$$(11) \quad \exp\left(O\left(\frac{\log x \log \log \log x}{\log \log x}\right)\right)$$

ways. Thus, the total number of choices for  $n$  in  $\mathcal{N}_5$  is

$$(12) \quad \begin{aligned} \#\mathcal{N}_5 &\leq \exp\left(O\left(\frac{\log x}{y} + \frac{\log x \log y}{\log \log x}\right)\right) \\ &= \exp\left(O\left(\frac{\log x \log \log \log x}{\log \log x}\right)\right). \end{aligned}$$

Hence, from estimates (7) and (12) we get

$$(13) \quad \#\mathcal{N}_3 \leq \#\mathcal{N}_4 + \#\mathcal{N}_5 \leq x^{O(\log \log \log x / \log \log x)}.$$

From estimates (4), (5) and (13), we finally get

$$\#\mathcal{N} \leq \#\mathcal{N}_1 + \#\mathcal{N}_2 + \#\mathcal{N}_3 \leq x^{O(\log \log \log x / \log \log x)},$$

which completes the proof of the theorem.

**3.2. Proof of Theorem 2.** We assume that  $y$  is as large as we wish otherwise there is nothing to prove. Let  $s = \lfloor 3 \log \log y \rfloor$ . Let

$$\mathcal{N} = \{n \in (x, x+y) : n \text{ has } k+j \text{ prime factors in } [A, B], 0 \leq j \leq s-1\},$$

with the parameters  $A = k^2, B = y^{1/(k+s+1)}$ , where we take  $k \in [c_1 \sqrt{\log y}, c_2 \sqrt{\log y}]$ , and  $0 < c_1 < c_2$  are two constants to be made more precise later. We will spend some time getting a lower bound on the cardinality of  $\mathcal{N}$ . For this, observe that for each  $n \in \mathcal{N}$  there is a squarefree number  $m$  with exactly  $k$  distinct prime factors in  $[A, B]$  such that  $m \mid n$ . Clearly,  $m \leq y^{k/(k+s+1)}$ . Fix such an  $m$  and put  $\mathcal{N}_m$  for the set of multiples of  $m$  in  $\mathcal{N}$ . To get a lower bound on  $\#\mathcal{N}_m$ , observe first that the number of multiples of  $m$  in  $(x, x+y)$  is

$$\geq \left\lfloor \frac{y}{m} \right\rfloor \geq \frac{y}{m} - 1 = \frac{y}{m} \left(1 + O\left(\frac{m}{y}\right)\right) = \frac{y}{m} \left(1 + O\left(\frac{1}{\log y}\right)\right).$$

Of course, not all such numbers are in  $\mathcal{N}_m$  since some of them might have more than  $k+s-1$  distinct prime factors in  $[A, B]$ . We next get an upper bound for the number of such "bad" multiples  $n$  of  $m$ . For each such bad  $n$ , there exists a number  $m_1$  having  $s$  prime factors in  $[A, B]$  and coprime to  $m$  such that  $mm_1 \mid n$ . Note that  $mm_1 \leq$

$y^{(k+s)/(k+s+1)} < y$ . For fixed  $m$  and  $m_1$ , the number of such positive integers in  $(x, x + y)$  is

$$\leq \left\lfloor \frac{y}{mm_1} \right\rfloor + 1 \leq \frac{2y}{mm_1}.$$

Summing up over all possibilities for  $m_1$ , we get that the number of such  $n$  is

$$\leq \frac{2y}{m} \sum_{\substack{m_1 \in \mathcal{M}_{s,A,B} \\ (m_1, m)=1}} \frac{1}{m_1} \leq \frac{2y}{ms!} \left( \sum_{A \leq p \leq B} \frac{1}{p} \right)^s = \frac{2yS^s}{ms!},$$

where we put

$$S := \sum_{A \leq p \leq B} \frac{1}{p}.$$

Observe that, by Mertens's formula, we have

$$\begin{aligned} S &= (\log \log B + c_0) - (\log \log A + c_0) + O\left(\frac{1}{\log A}\right) \\ &= \log \log y - \log(k + s + 1) - \log \log k - \log 2 + O\left(\frac{1}{\log k}\right) \\ &= \log \log y - \log k - \log \log k - \log 2 + O\left(\frac{1}{\log k} + \frac{s}{k}\right). \end{aligned}$$

As far as errors go, note that since  $s = 3 \log \log y + O(1)$  and  $k \asymp \sqrt{\log y}$ , we have that

$$\frac{s}{k} \ll \frac{\log k}{k} \ll \frac{1}{\log k}.$$

Furthermore,  $S = (1/2 + o(1)) \log \log y$  as  $y \rightarrow \infty$ , therefore for  $y > y_0$  we have that  $S < s/3$ . We record that

$$(14) \quad S = \log \log y - \log k - \log \log k - \log 2 + O\left(\frac{1}{\log k}\right).$$

The above arguments show that

$$\#\mathcal{N}_m \geq \frac{y}{m} \left( 1 - \frac{2S^s}{s!} + O\left(\frac{1}{\log y}\right) \right).$$

From the elementary estimate  $s! \geq (s/e)^s$ , we get

$$\frac{2S^s}{s!} \ll \left(\frac{Se}{s}\right)^s \ll \left(\frac{e}{3}\right)^s$$



and the last number above is  $< 1/3$  if  $y$  is sufficiently large. Hence, the inequality

$$\#\mathcal{N}_m \geq \frac{y}{2m}$$

holds uniformly in squarefree integers  $m$  having  $k$  distinct prime factors all in  $[A, B]$ . We now sum over  $m$  and use Lemma 4 to get that

$$(15) \quad \sum_{m \in \mathcal{M}_{k,A,B}} \#\mathcal{N}_m \geq \frac{y}{2} \sum_{m \in \mathcal{M}_{k,A,B}} \frac{1}{m} \gg \frac{yS^k}{k!} \left( 1 + O\left(\frac{k^2}{S^2 A \log A}\right) \right) \gg \frac{yS^k}{k!}$$

for large  $y$ , because  $A = k^2$ , therefore the expression  $k^2/(S^2 A \log A)$  is arbitrarily small if  $y$  is large. Next let us note that if  $n \in \mathcal{N}$ , then  $n$  has  $k+j$  distinct prime factors in  $[A, B]$  for some  $j = 0, 1, \dots, s-1$ . Thus, the number of possibilities for  $m \mid n$  in  $\mathcal{M}_{k,A,B}$  is

$$\binom{k+j}{k} \leq \binom{k+s}{s} < \left(e + \frac{ek}{s}\right)^s = \exp(O((\log \log y)^2)).$$

Here, we used again the fact that  $s! \geq (s/e)^s$ . In particular, the sum on the left of (15) counts numbers  $n \in \mathcal{N}$  and each number is counted at most  $\exp(O((\log \log y)^2))$  times. Hence, dividing by this number we get a lower bound on  $\#\mathcal{N}$  which is

$$\#\mathcal{N} \geq \frac{yS^k}{k!} \exp(O((\log \log y)^2)).$$

If  $n \in \mathcal{N}$ , then there is an  $m \in \mathcal{M}$  such that  $m \mid n$ . It now follows, from Lemma 2, that  $f(n) \geq f(m) \geq B_k$ . Thus,

$$\frac{1}{y} \sum_{x \leq n \leq x+y} f(n) \geq \frac{1}{y} \sum_{n \in \mathcal{N}} f(n) \geq \frac{1}{y} B_k \#\mathcal{N} \geq \frac{B_k S^k}{k!} \exp(O((\log \log y)^2)).$$

We now maximize  $B_k S^k/k!$  by choosing  $k$  appropriately versus  $y$ . Using Stirling's formula

$$k! \sim \left(\frac{k}{e}\right)^k (2\pi k)^{1/2}$$

to estimate  $k!$ , Lemma 1 as well as estimate (14), we get

$$(16) \quad \frac{B_k S^k}{k!} \exp(O((\log \log y)^2)) = \exp\left(h(k) + O\left(\frac{k(\log \log k)^2}{(\log k)^2}\right)\right),$$

where the function  $h(k)$  is

$$\begin{aligned} h(k) &= k \log(\log \log y - \log k - \log \log k - \log 2) \\ &\quad - k \log \log k + k \frac{\log \log k}{\log k} + \frac{k}{\log k}. \end{aligned}$$

The error term under the exponential in formula (16) comes from the estimate given by Lemma 1 on  $B_k$ , estimate (14) which tells us that

$$\begin{aligned} k \log S &= k \log \left( \log \log y - \log k - \log \log k - \log 2 + O \left( \frac{1}{\log k} \right) \right) \\ &= k \log(\log \log y - \log k - \log \log k - \log 2) + O \left( \frac{k}{(\log k)^2} \right), \end{aligned}$$

because  $\log \log y - \log k - \log \log k - \log 2 \asymp \log k$  for our choice of  $k$  versus  $y$ , as well as the fact that  $(\log \log y)^2 \ll k(\log \log k)^2/(\log k)^2$ , again by our choice of  $k$  versus  $y$ .

We now choose

$$k = \left\lfloor \frac{1}{\sqrt{2e}} (\log y)^{1/2} \right\rfloor.$$

Note that with  $c_1 = 1/4$  and  $c_2 = 1/2$  we indeed have that  $k \in [c_1(\log y)^{1/2}, c_2(\log y)^{1/2}]$ , as promised. Then,

$$\begin{aligned} k &= \frac{1}{\sqrt{2e}} (\log y)^{1/2} + O(1); \\ \log k &= \frac{1}{2} \log \log y - \log(\sqrt{2e}) + O \left( \frac{1}{\sqrt{\log y}} \right); \\ \frac{1}{\log k} &= \frac{2}{\log \log y} + O \left( \frac{1}{(\log \log y)^2} \right). \end{aligned}$$

In particular,

$$\begin{aligned} \log \log y &- \log k - \log \log k - \log 2 \\ &= \frac{1}{2} \log \log y + \log(\sqrt{2e}/2) - \log \log k + O \left( \frac{1}{\sqrt{\log y}} \right) \\ &= \left( \frac{1}{2} \log \log y - \log(\sqrt{2e}) \right) - \log \log k + 1 + O \left( \frac{1}{\sqrt{\log y}} \right) \\ &= \log k - (\log \log k - 1) + O \left( \frac{1}{\sqrt{\log y}} \right), \end{aligned}$$

so that

$$\begin{aligned} \log(\log \log y &- \log k - \log \log k - \log 2) \\ &= \log \left( \log k - (\log \log k - 1) + O \left( \frac{1}{\sqrt{\log y}} \right) \right) \\ &= \log(\log k - (\log \log k - 1)) + O \left( \frac{1}{k \sqrt{\log y}} \right) \\ &= \log(\log k - (\log \log k - 1)) + O \left( \frac{1}{\log y} \right). \end{aligned}$$

Thus,

$$\begin{aligned}
k \log(\log \log y - \log k - \log \log k - \log 2) - k \log \log k \\
&= k \log \left( \frac{\log k - (\log \log k - 1)}{\log k} \right) \left( 1 + O \left( \frac{1}{\log y} \right) \right) \\
&= k \log \left( 1 - \frac{\log \log k - 1}{\log k} \right) + O \left( \frac{1}{\sqrt{\log y}} \right) \\
&= -\frac{k(\log \log k - 1)}{\log k} + O \left( \frac{k(\log \log k)^2}{(\log k)^2} + \frac{1}{k} \right) \\
&= -\frac{k \log \log k}{\log k} + \frac{k}{\log k} + O \left( \frac{k(\log \log k)^2}{(\log k)^2} \right).
\end{aligned}$$

It now follows immediately that

$$\begin{aligned}
h(k) &= k \log(\log \log y - \log k - \log \log k - \log 2) - k \log \log k \\
&\quad + \frac{k \log \log k}{\log k} + \frac{k}{\log k} = \frac{2k}{\log k} + O \left( \frac{k(\log \log k)^2}{(\log k)^2} \right).
\end{aligned}$$

One can in fact check that the above estimate is the maximum of  $h(k)$  as a function of  $k$  when  $y$  is fixed. We will not drag the reader through this computation. Comparing the above estimate with (16), we get that

$$\begin{aligned}
\frac{B_k S^k}{k!} \exp(O((\log \log y)^2)) &\geq \exp \left( \frac{2k}{\log k} + O \left( \frac{k(\log \log k)^2}{(\log k)^2} \right) \right) \\
&= \exp \left( \frac{4}{\sqrt{2e}} \frac{(\log y)^{1/2}}{\log \log y} \left( 1 + O \left( \frac{(\log \log \log y)^2}{\log \log y} \right) \right) \right).
\end{aligned}$$

We thus get that

$$\begin{aligned}
\frac{1}{y} \sum_{x \leq n \leq x+y} f(n) &\geq \frac{B_k S^k}{k!} \exp(O((\log \log y)^2)) \\
&\geq \exp \left( \left( \frac{4}{\sqrt{2e}} + O \left( \frac{(\log \log \log y)^2}{\log \log y} \right) \right) \frac{\sqrt{\log y}}{\log \log y} \right),
\end{aligned}$$

which is what we wanted.

**3.3. Proof of Theorem 3.** We observe that primes are in  $\mathcal{A}$  as  $f(p) = 1$  for all prime  $p$ . Thus,

$$\#\mathcal{A}(x) \gg \frac{x}{\log x}.$$

This completes the lower bound part of the theorem. To obtain the upper bound, we cover the set  $\mathcal{A}(x)$  by three subsets  $\mathcal{A}_1$ ,  $\mathcal{A}_2$  and  $\mathcal{A}_3$

as follows:

$$\mathcal{A}_1 = \{n \leq x : \Omega(n) > 10 \log \log x\},$$

$$\mathcal{A}_2 = \left\{n \leq x : \omega(n) < \frac{\log \log x}{\log \log \log x}\right\},$$

and

$$\mathcal{A}_3 = \{n \leq x : n \equiv 0 \pmod{f(n)}, n \notin \mathcal{A}_1 \cup \mathcal{A}_2\}.$$

We recall the following bound

$$\#\{n \leq x : \Omega(n) = k\} \ll \frac{kx}{2^k}$$

valid uniformly in  $k$  (see, for example, Lemma 13 in [8]). Using the above estimate, we get

$$(17) \quad \#\mathcal{A}_1 \leq x \sum_{k > 10 \log \log x} \frac{k}{2^k} \ll \frac{x \log \log x}{2^{10 \log \log x}} = o\left(\frac{x}{\log x}\right)$$

as  $x \rightarrow \infty$ . To find an upper bound for  $\mathcal{A}_2$ , we use the Hardy-Ramanujan bounds (see [6])

$$\#\{n \leq x : \omega(n) = k\} \ll \frac{x (\log \log x + c_1)^{k-1}}{\log x (k-1)!}$$

with some positive constant  $c_1$ . Using the elementary estimate  $m! \geq (m/e)^m$  with  $m = k-1$ , we get

$$\#\{n \leq x : \omega(n) = k\} \ll \frac{x}{\log x} \left( \frac{e \log \log x + c_2}{k-1} \right)^{k-1},$$

where  $c_2 = ec_1$ . The right hand side is an increasing function of  $k$  in our range for  $k$  versus  $x$  when  $x$  is large. Since  $k < (\log \log x)/(\log \log \log x)$ , we deduce that

$$(18) \quad \#\mathcal{A}_2 \ll \frac{x}{\log x} (O(\log \log \log x))^{\log \log x / \log \log \log x} = \frac{x}{(\log x)^{1+o(1)}}$$

as  $x \rightarrow \infty$ .

Finally, we estimate  $\mathcal{A}_3$ . Each  $n \in \mathcal{A}_3$  can be written as

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_k^{\alpha_k},$$

where  $q_1, \dots, q_k$  are distinct primes,  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k$ ,  $\alpha_1 + \alpha_2 + \dots + \alpha_k \leq 10 \log \log x$  and  $k > K := \lfloor \log \log x / \log \log \log x \rfloor$ . Let  $\mathcal{T}$  be the set of all such tuples  $(k, \alpha_1, \dots, \alpha_k)$ . For each such  $n$ , we have that

$$\begin{aligned} f(n) &\geq B_K \geq \exp((1+o(1))K \log K) \geq \exp((1+o(1)) \log \log x) \\ &= (\log x)^{1+o(1)}. \end{aligned}$$

The number of tuples  $(k, \alpha_1, \dots, \alpha_k)$  satisfying the above conditions is at most

$$\#\mathcal{T} \ll \log \log x \sum_{n \leq 10 \log \log x} p(n),$$

where again  $p(n)$  is the partition function of  $n$ . Using estimate (2), we get that the cardinality of  $\mathcal{T}$  is at most

$$\#\mathcal{T} \ll (\log \log x)^2 \exp(O(\sqrt{\log \log x})) = (\log x)^{o(1)} \quad \text{as } x \rightarrow \infty.$$

Thus,

$$(19) \quad \#\mathcal{A}_3 \ll \sum_{(k, \alpha_1, \dots, \alpha_k) \in \mathcal{T}} \frac{x}{f(p_1^{\alpha_1} \dots p_k^{\alpha_k})} \ll \frac{x \#\mathcal{T}}{B_K} = \frac{x}{(\log x)^{1+o(1)}}$$

as  $x \rightarrow \infty$ . Now inequalities (17), (18) and (19) yield the desired upper bound and complete the proof.

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